Bridging Linearized General Relativity and Quantum Gravity

D Hidalgo, N Inan

Clovis Community College

ΨΦΜΛ Society of Physics and Math Legends

BACKGROUND

Gravitational waves in terms of electric-like and magnetic-like tensor fields

In GR, writing the metric as a perturbation to flat space-time gives $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. For weak gravitational fields the only propagating degrees of freedom in linearized GR are the transverse-traceless spatial perturbations h_{ij}^{TT}, where $\partial_i h_{ij}^{TT} = 0$ and $\eta^{ij} h_{ij}^{TT} = 0$. [1]
A gravito-electric tensor field and a gravito-magnetic tensor field can be defined respectively as [2]

$$E_{ij} \equiv -\partial_t h_{ij}^{TT} \quad \text{and} \quad B_{ij} \equiv \varepsilon_{ikl}\partial_k h_{lj}^{TT} \quad (1)$$

These fields are analogous to $\vec{E} = -\partial_t \vec{A}$ and $\vec{B} = \nabla \times \vec{A}$ in electromagnetism (EM).

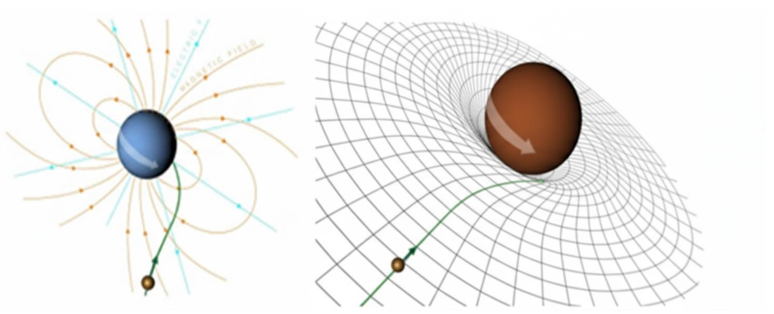

FIGURE 1: A test particle in an EM field (left) versus a gravitational field (right). Equations of motion are $a^i = \frac{q}{m}\left(E^i + \varepsilon_{ijk} v^j B^k\right)$ for EM and $a^i = v^j E_{ij} + \frac{1}{4} v_j v_k \left(\varepsilon_{ijk} B_{lk} + 2\varepsilon_{ikl} B_{lj}\right)$ for GR.

The Isaacson Power Flux Formula

The Isaacson power flux formula describes the intensity of gravitational waves in GR as

$$I_{\mathrm{GR}} = \frac{1}{4c\kappa}\left\langle \left(\dot{h}_{ij}^{TT}\right)^2 \right\rangle \quad (2)$$

where $\kappa \equiv 8\pi G/c^4$. [3] Using (1), we can write (2) as $I_{\mathrm{GR}} = \frac{1}{4c\kappa}\left\langle E_{ij}^2 \right\rangle$. Note that $\left\langle E_{ij}^2 \right\rangle = \frac{1}{2} E_{ij,0}^2$, where $E_{ij,0}$ is the tensor amplitude of a sinusoidal field. Therefore, (2) becomes

$$I_{\mathrm{GR}} = \frac{1}{8c\kappa} E_{ij,0}^2 \quad (3)$$

analogous to $I_{\mathrm{EM}} = \frac{1}{2c\mu_0} E_0^2$ in EM, where E_0 is electric field amplitude.

Quantizing the Gravitational Wave Perturbation

The classical field can be promoted to a quantum operator as

$$\hat{h}_{ij}^{\mathrm{TT}}(\mathbf{r},t) = L_{\mathrm{Pl}} \sum_{\mathbf{k}} \sum_{p=1}^{2} \sqrt{\frac{4\pi c}{\tilde{V}\omega_{\mathbf{k}}}} \left[\xi_{ij,\mathbf{k},p} \hat{b}_{\mathbf{k}} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)} + \mathrm{h.c.}\right] \quad (4)$$

where $L_{\mathrm{Pl}} = \sqrt{G\hbar/c^3}$ is the Planck length, $\tilde{V}$ is the mode volume, $\mathbf{k}$ is the wave vector, and ξ_{ij} are the two transverse-traceless (TT) polarization tensors of a gravitational wave. Also, $\hat{b}$ and $\hat{b}^\dagger$ are graviton annihilation and creation operators, respectively. [4, 5]

ABSTRACT

The classical energy of gravitational waves is amplitude dependent, while the quantum energy of gravitons is frequency dependent. Applying the Bohr Correspondence Principle (BCP) to the case of gravity predicts that the classical and quantum theories should be consistent in the limit of large quantum numbers (i.e. numerous gravitons). A formal treatment is shown using linearized General Relativity (GR) and assuming the quantization of the gravitational field. Furthermore, using quantized gravito-electric and gravito-magnetic tensor fields, we obtain a canonical quantization of the gravitational field and an associated uncertainty principle.

MAIN POINTS

Quantizing Gravitational Wave Tensor Fields

Using (1) and (4) leads to quantized fields given respectively as

$$\hat{E}_{ij}(\mathbf{r},t) = i\sum_{\mathbf{k}} \sum_{p=1}^{2} \sqrt{\frac{4\pi G\hbar\omega_{\mathbf{k}}}{c^2\tilde{V}}} \left[\xi_{ij,\mathbf{k},p} \hat{b}_{\mathbf{k}} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)} + \mathrm{h.c.}\right] \quad (5)$$

and

$$\hat{B}_{ij}(\mathbf{r},t) = i\sum_{\mathbf{k}} \sum_{p=1}^{2} \sqrt{\frac{4\pi G\hbar}{c^2\tilde{V}\omega_{\mathbf{k}}}} \varepsilon_{ilm} k_l \left[\xi_{mj,\mathbf{k},p} \hat{b}_{\mathbf{k}} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)} + \mathrm{h.c.}\right] \quad (6)$$

These newly defined fields are analogous to the quantized EM fields, $\hat{\boldsymbol{E}}$ and $\hat{\boldsymbol{B}}$ fields.

Canonical Quantization of the Graviton Field

Lagrangian density for the gravitational field is [5]

$$\mathcal{L} = \frac{1}{16c^2\kappa}\left(E_{ij}^2 - c^2 B_{ij}^2\right)$$

The canonical momentum tensor density is found as

$$\wp^{ij} = \frac{\partial\mathcal{L}}{\partial\dot{h}_{ij}^{\mathrm{TT}}} = -\frac{1}{8c^2\kappa} E_{ij}$$

Using $p^{ij} = \tilde{V}\wp^{ij}$, promoting to a quantum operator and applying (5) leads to

$$\hat{p}_{ij} = -\frac{ic^2}{32}\sum_{\mathbf{k}} \sum_{p=1}^{2} \sqrt{\frac{\hbar\omega_{\mathbf{k}}}{\pi G\tilde{V}}} \left[\xi_{ij,\mathbf{k},p} \hat{b}_{\mathbf{k}} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)} + \mathrm{h.c.}\right] \quad (7)$$

The associated Hamiltonian density operator is

$$\hat{\mathcal{H}} = \frac{1}{16c^2\kappa}\left(\hat{E}_{ij}^2 + c^2\hat{B}_{ij}^2\right)$$

The operators in (4) and (7) satisfy the following commutation relations.

$$\left[\hat{h}_{ij,\boldsymbol{k}}^{\mathrm{TT}}, \hat{p}^{*ij}_{\boldsymbol{k'}}\right] = \left[\hat{h}^{*\mathrm{TT}}_{ij,\boldsymbol{k}}, \hat{p}^{ij}_{\boldsymbol{k'}}\right] = \frac{i\hbar}{8}\delta_{\boldsymbol{k},\boldsymbol{k'}}$$

The generalized uncertainty principle is given by $\left(\frac{1}{2i}\left\langle\left[\hat{A},\hat{B}\right]\right\rangle\right)^2 \le \left(\frac{\hbar}{2}\right)^2$, where $\hat{A}$ and $\hat{B}$ are canonical conjugate operators. [7] From this we obtain

$$\boxed{\Delta h_{\boldsymbol{k}}^{\mathrm{TT}}\,\Delta p_{\boldsymbol{k}} \ge \frac{\hbar}{16}}$$

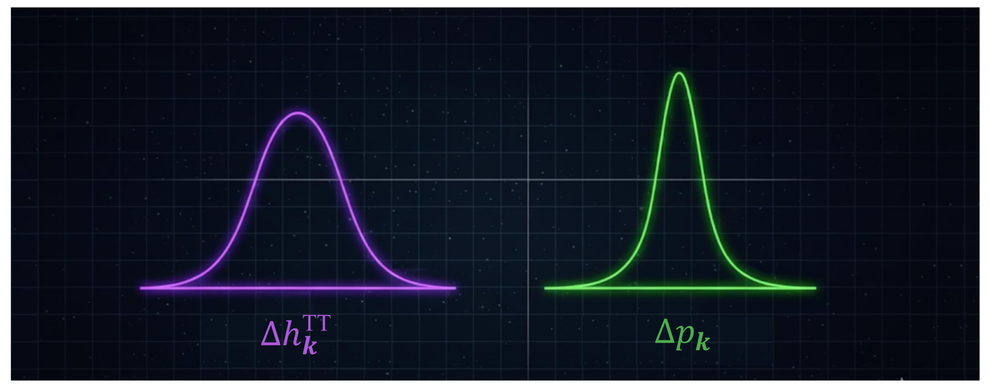

FIGURE 2: Example of distributions for the canonical conjugate pair: $h_{\boldsymbol{k}}^{\mathrm{TT}}$ and $p_{\boldsymbol{k}}$.

Isaacson Power from Quantized Linear Gravity

To find the energy density, we use (5) and (6) and take the modulus squared, $\hat{E}_{ij}^2 = \hat{E}_{ij}^*\hat{E}^{ij}$, to obtain

$$\hat{E}_{ij}^2 = \frac{4\kappa\hbar c^2}{\tilde{V}}\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left[2N_{\mathbf{k}} + 1 - \hat{b}_{\mathbf{k}}^{\mathrm{T}}\hat{b}_{\mathbf{k}} e^{2i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)} + \mathrm{h.c.}\right] \quad (8)$$

and

$$\hat{B}_{ij}^2 = \frac{4\kappa\hbar}{\tilde{V}}\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left[2N_{\mathbf{k}} + 1 + \hat{b}_{\mathbf{k}}^{\mathrm{T}}\hat{b}_{\mathbf{k}} e^{2i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)} + \mathrm{h.c.}\right] \quad (9)$$

where N is graviton number. Combining (8) and (9) gives

$$\langle u_{\mathrm{GR}}\rangle = \hat{E}_{ij}^2 + c^2\hat{B}_{ij}^2 = \frac{16\kappa\hbar c^2}{\tilde{V}}\sum_{\boldsymbol{k}}\omega_{\boldsymbol{k}}\left(N_{\boldsymbol{k}} + \frac{1}{2}\right) \quad (10)$$

analogous to $\langle u_{\mathrm{EM}}\rangle = \varepsilon_0\hat{\boldsymbol{E}}^2 + \frac{1}{\mu_0}\hat{\boldsymbol{B}}^2 = \frac{2\hbar}{\tilde{V}}\sum_{\boldsymbol{k}}\omega_{\boldsymbol{k}}\left(N_{\boldsymbol{k}} + \frac{1}{2}\right)$ [8].
Gravitational intensity, $I_{\mathrm{GR}} = c\langle u_{\mathrm{GR}}\rangle$, is therefore

$$\boxed{I_{\mathrm{GR}} = \frac{1}{8\kappa c}\hat{E}_{ij}^2 = \frac{\hbar}{\tilde{V}}\sum_{\boldsymbol{k}}\omega_{\boldsymbol{k}}\left(N_{\boldsymbol{k}} + \frac{1}{2}\right)}$$

This matches (3), hence the Isaacson power flux is the sum over graviton modes. For large graviton number, we obtain a continuous spectrum which demonstrates the BCP.

$$I_{\mathrm{GR}} = \frac{1}{8\kappa c}E_{ij}^2 = \frac{\hbar}{\pi^2 c^2}\int\left(N_\omega + \frac{1}{2}\right)\omega^3 d\omega$$

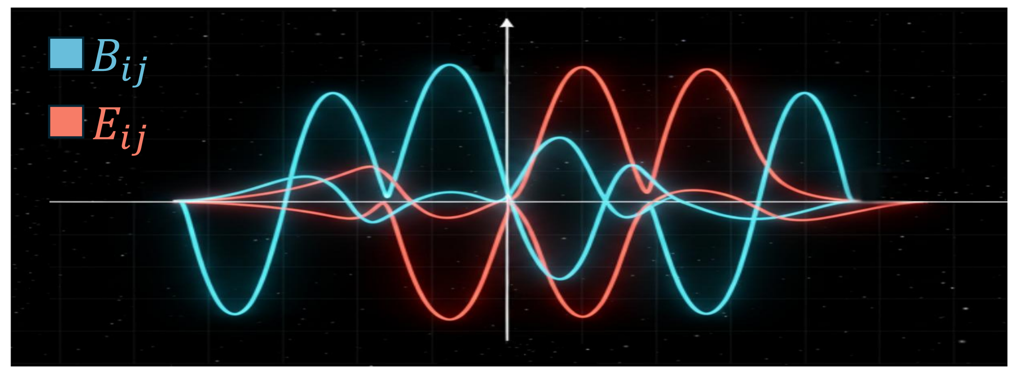

FIGURE 3: Visualization of E_{ij} and B_{ij} oscillations for a linearized gravitational wave. The smooth waveform emerges from the continuum limit of quantized graviton modes.

Relating Classical and Quantum Field Amplitudes

Applying (10) to a single classical mode leads to

$$E_{ij,0}^2 = c^2 B_{ij,0}^2 = 8c^2\kappa\frac{N\hbar\omega}{V} \quad (11)$$

The mode pre-factors for the quantum fields in (4) – (6) are

$$\hat{E}_{ij,0} = c\hat{B}_{ij,0} = \sqrt{\frac{4G\hbar\pi\omega}{c^2\tilde{V}}} \quad (12)$$

Using (11) and (12), and taking the limit of large graviton number (so that $\tilde{V} = V$) leads to

$$\boxed{E_{ij,0} = 4\sqrt{N}\hat{E}_{ij,0} \quad \text{and} \quad B_{ij,0} = 4\sqrt{N}\hat{B}_{ij,0}}$$

These relations between classical and quantum amplitudes are further demonstrations of the BCP.

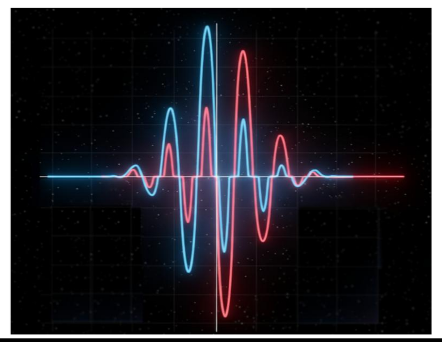

FIGURE 4: A localized graviton wave packet formed by a superposition of quantum modes, $\hat{E}_{ij}$ and $\hat{B}_{ij}$. For large N, graviton wave packets add coherently to produce the classical fields, E_{ij} and B_{ij}.

CONCLUSIONS

Quantizing gravitational wave tensor fields

From the quantized perturbation in linearized GR, we obtained quantized electric-like and magnetic-like tensor fields for gravitational waves.

Canonical Quantization of the Graviton Field

The Lagrangian for gravitational waves in linearzed GR was used to obtain a canonical momentum that is the conjugate to the quantized perturbation. This leads to a quantized Hamiltonian density and an uncertainty relation for the graviton field.

Isaacson power from quantized linear gravity

The classical expression for gravitational wave intensity was formally obtained from the quantized perturbation field in the limit of large graviton number. This is a demonstration of the Bohr Correspondence Principle applied to gravitational waves.

Relating classical and quantum field amplitudes

The *classical* amplitude of gravitational waves and the *quantum* amplitude of the field operators are related by the graviton number in the limit of numerous gravitons. This provides another manifestation of the Bohr Correspondence Principle in the context of quantum gravity.

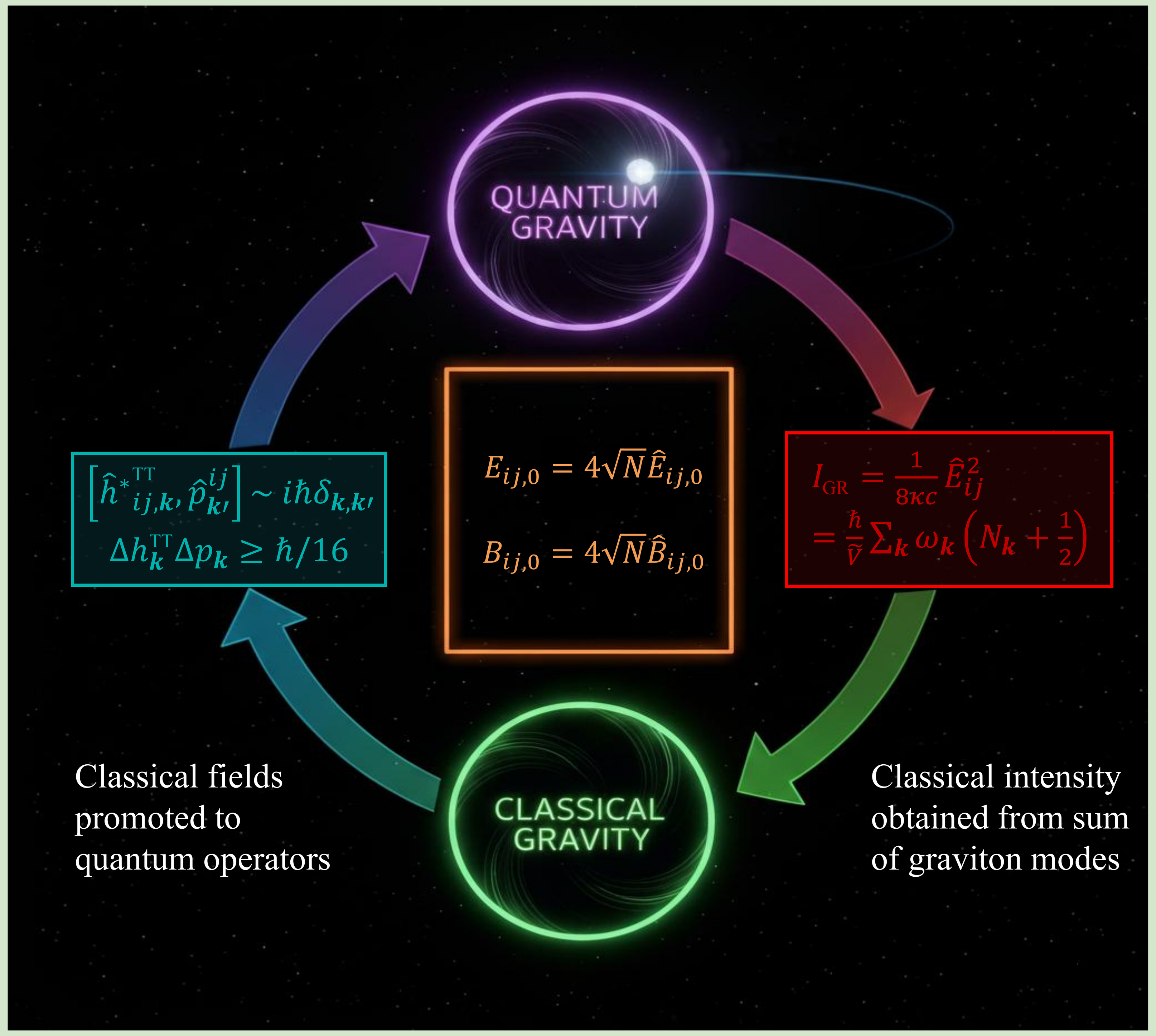

FIGURE 5: Visualization of key relationships connecting classical gravitational waves to quantum gravitons in a model of the BCP applied to gravity

References

1. Flanagan, Hughes (2005); 2. Inan (2016, 2018); 3. Misner, Thorne, Wheeler (1972); 4. Grishchuk (1974, 1976, 1992, 2001, 2005); 5. Carney (2024); 6. Barnett (2014); 7. Griffiths (1981); 8. Inan, Ghosoph, Hidalgo, et. al. (in prep); 9. Bohr (1918-1923)